Anyons : ~~Part 1~~ :

One of the most important concepts in many particle systems is that of indistinguishable particles. The idea of this is that for identical particles, exchange of any 2 particles is a symmetry operation. In 3-D space, this leads to either bosons or fermions. ~~But in 2D~~ But in 2D, a variety of statistics — any statistics — is possible and we don't even have a complete classification of all possibilities.

~~What was initially called anyons were just~~

Anyons first came into prominence when it was realised that quasi-particle excitations in the FQHE could have fractional statistics. There was also a time when people thought that high T_c superconductivity could be explained by anyons. But at the current time, the reason that there is a lot of interest in anyons is because quantum computation. For this, you actually need more complicated anyons which obey non-abelian statistics, but right now, I will first start with ~~the~~ the simpler abelian anyons.

Let me start by explaining why you can get anyons only in 2 Dimension. ~~The~~

In 3 dimensions, let us consider the statistics of 2 identical particles. The configuration space is given by $(\vec{r}_1, \vec{r}_2)$. Indistinguishability means that

$(\vec{r}_1, \vec{r}_2) \sim (\vec{r}_2, \vec{r}_1)$ where position (1, 2) $\Rightarrow$ particle 1 & 2.

$(\vec{r}_1, \vec{r}_2) \Rightarrow$ first particle at $\vec{r}_1$
second " " $\vec{r}_2$

$(\vec{r}_2, \vec{r}_1) \Rightarrow$ first particle at $\vec{r}_2$
second " " $\vec{r}_1$

Also impose $\vec{r}_1 = \vec{r}_2$ (hard-core constraint)

Then in relative & CM co-ordinates, configuration space is $(\vec{R}, \vec{r})$

$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$ $\quad \vec{r} = \vec{r}_1 - \vec{r}_2$.

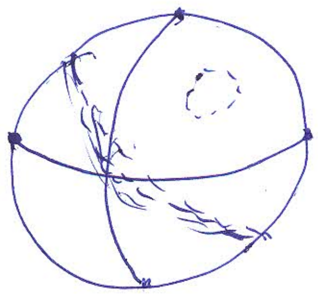
Statistics only affects $\vec{r}$. $\vec{r} \equiv -\vec{r}$
for indistinguishable ptcles.

The space is thus

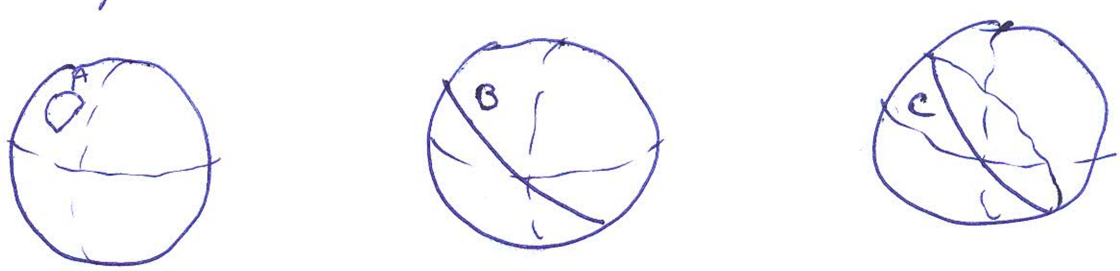
$$R_3 \times \underbrace{\frac{R_3 - \text{origin}}{Z_2}}$$

So we need to classify closed paths in $\frac{R_3 - \text{origin}}{Z_2}$

If we keep $|\vec{r}|$ fixed - The tip of $\vec{r}$ defines the surface of a sphere. $\vec{r} \equiv -\vec{r} \Rightarrow$ opposite points on the sphere are identified



Here, the ~~ofa~~ phase picked up by the wave-function of the 2 particles under exchange can only fall in 2 classes. Either trivial or it can go from one edge to the other – i.e. it can have phase 1 or -1. All other possibilites can be reduced to these 2 because path C can be deformed to path A



Two exchanges are equivalent to no exchanges $\Longrightarrow$ phase $\eta^2 = 1$
$\Longrightarrow \eta = \pm 1$.

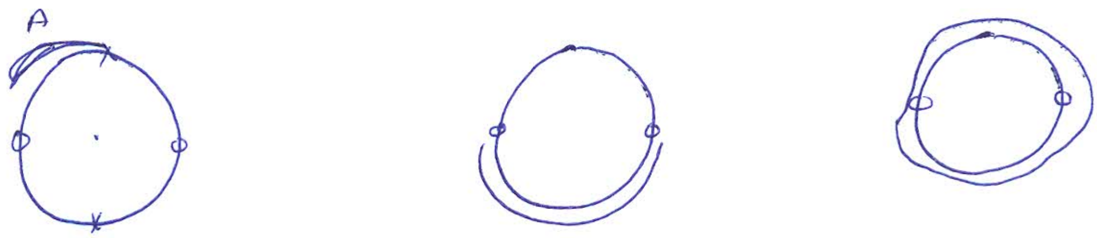
This leads to fermions & bosons.

Why does this argument break down in 2+1 dimensions.

Here configuration space is given by

$$R_2 \otimes \frac{R_2 - \text{origin}}{Z_2}$$

Here, there are several possible closed paths (around a O.)

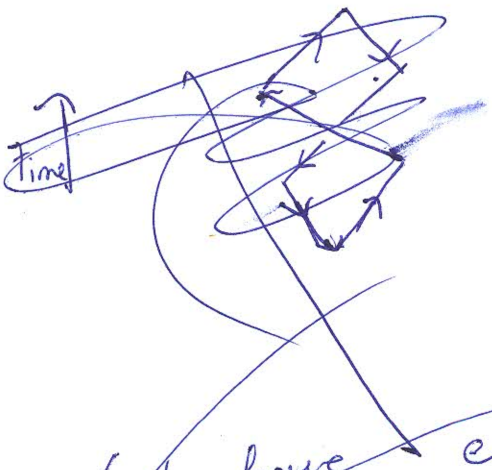


Explain permutation & braid a little here.

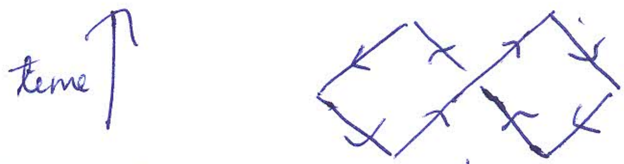
The ~~point~~ path that involves no exchanges can be shrunk to a point but all the other paths are non-contractible. Thus if η = phase under single exchange, one can have η^2, η^3, etc. Since the modulus of the wave-fn remains unchanged, all we can say is that it is a phase $e^{i\theta}$.

that the particle and anti-ptcle together will have trivial quantum numbers and the particle and the anti-ptcle can be pair created.

Let us now draw the world-line of ~~too~~ two ptcle-anti-ptcle pair, exchange the ptcle ~~and then annihilate it.~~ from one pair with the particle from the other pair and then annihilate both the pairs.

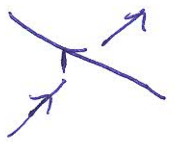


(I have exchanged



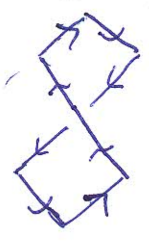
this gives a particular sense to the exchange (call it counter clock-wise)

(Then

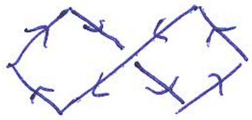


would be clock-wise)

This same diagram can be turned by 90° and interpreted as a single ptcle anti-ptcle pair, exchanged in clock-wise sense and then annihilated



We can also interpret it as a process where anti-prcles are exchanged by turning it again by 90° again in a counter clock-wise sense



By thinking of the particles as narrow ribbons, ~~this is~~ all these processes are equivalent to a counter clockwise twist of (a ribbon with a 2π)

So it is equivalent to a process where a particle rotates thru' 2π $\cancel{e^{i2\pi J}}$ $\cancel{e^{i2\pi J}}$ it acquires — i.e. a phase $e^{-i\theta J} = e^{-i2\pi J}$

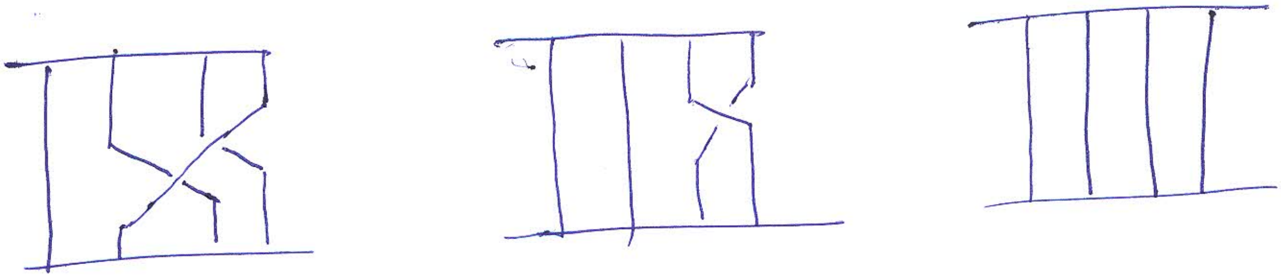
In terms of the exchanges. the phase that is generated by rotating a prcle counter-clockwise by 2π is equivalent to the phase obtained by exchanging 2 prcles counter-clockwise or 2 anti-prcles counter-clockwise or exchanging a prcle and anti-prcle clock-wise. This is the spin-statistics connection for anyons.

Now, let us consider anyons as representations of the braid group. First, let me make the distinction between the braid group and the permutation group clear. ~~Because the phas~~ If we consider phases under just ~~exc~~ under exchange of quantum #'s of 2 ptcles, then they are representations of the permutation group P_N, but if we consider phases under an adiabatic exchange of ptcles, then they are representations of the braid group.

The permutation gp. (P_N) is the gp. formed by all possible permutations of N objects, with gp. multiplication just being successive permutations and gp inverse is undoing the permutation. Here, it is clear that the square of ~~the~~ any permutation is just 1 and the ptcles can only be fermions or bosons.

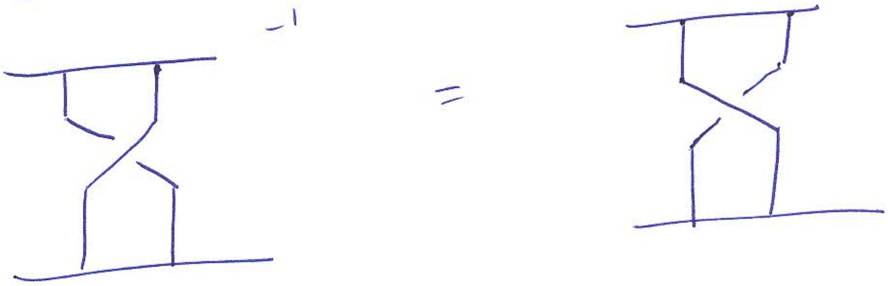
But the braid gp. B_N is the gp of inequivalent trajectories that occur when adiabatically ~~exchang~~ moving N objects.

For example, 

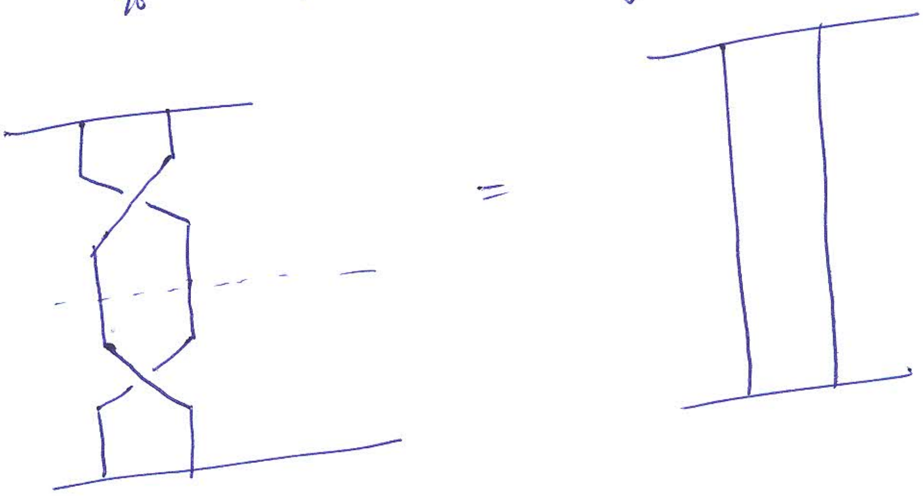


are all elements of B_4.

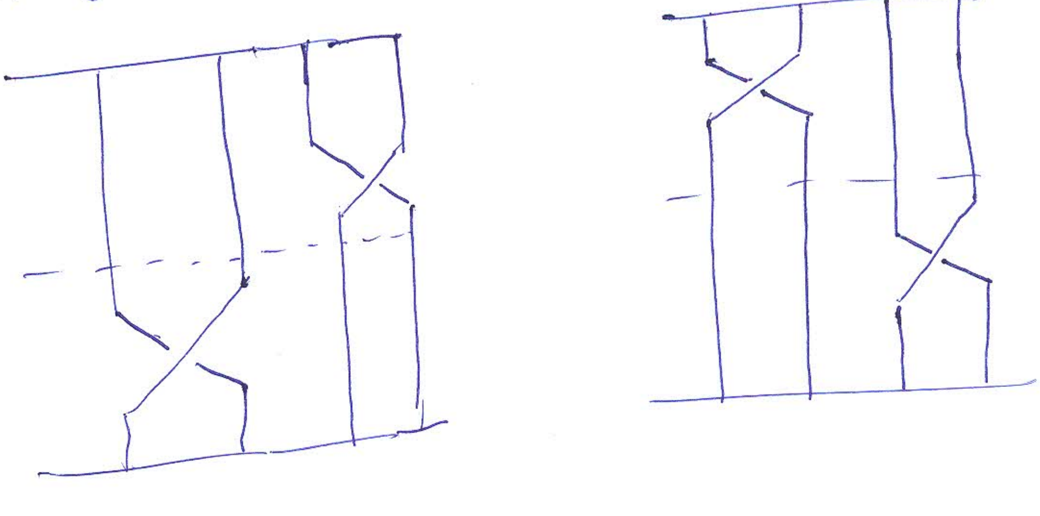
Group inverse is defined as

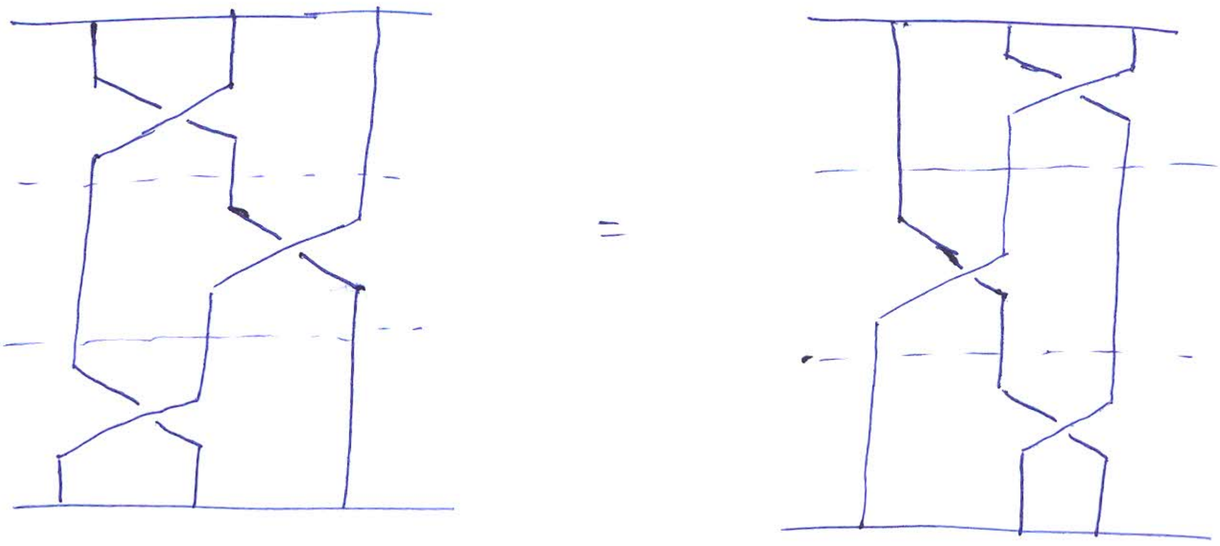


Product of trajectory and its inverse leads to the identity



Pictorially, we can also see that





Yang - Baxter relation.

Now, let us say the same thing a little more abstractly. Consider n ~~xx~~ indistinguishable point-like particles confined to a 2 dimensional surface. The config. of the particles at a fixed time corresponds to this plane with n punctures. If you perform a permutation of these punctures, it would have no physical effect. But to evaluate the quantum amplitude for this config to evolve to another config of n particles at time $t = T$, we need to sum over all paths keeping the initial & final configurations fixed, weighted by the action e^{iS}. If we think of the world-lines as threads, then each path (or history) becomes a braid where a particle at $t = 0$ can be connected to any of the ptcles at $t = T$ by a thread. Since ptcle world-lines cannot cross, the braids fall into distinct topological classes that cannot be smoothly deformed into one another.

Thus the elements of the gp. generated by adiabatic exchanges of n ptcles is isomorphic to the braid gp. of n strands.

In the quantum theory, the quantum state of n indistinguishable particles belongs to a Hilbert space that transforms as a unitary representation of the braid gp. B_n

The braid gp can be represented by a set of generators obeying the following –

Imagine n-ptcles occupying n-ordered positions on a line.

Let σ_1 denote a counter clock-wise exchange of ptcles occupying positions 1 & 2

σ_2 — exchange of ptcles occupying 2 & 3

σ_3 " " " " 3 & 4

etc.

Since any braid can be constructed by exchanging 2 nearby threads, $\sigma_1, \sigma_2 \; -- \; \sigma_{n-1}$ are the generators of the gp.

The relations satisfied by these generators are

$$\sigma_j \sigma_k = \sigma_k \sigma_j \qquad |j-k| \geq 2$$

and $\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}$, $j = 1 \ldots n-2$

The second is the Yang-Baxter relation.

Since the braid gp is ∞, it has an ∞ # of unitary irreducible repns. and also an infinite # of one-dimensional or abelian rep^{ns}. There are also non-abelian repns. which will be the next topic. But in the abelian rep^{ns}, each generator can be written as $\sigma_j = e^{i\theta_j}$ and the Yang-Baxter relation is simply

$$e^{i\theta_j} e^{i\theta_{j+1}} e^{i\theta_j} = e^{i\theta_{j+1}} e^{i\theta_j} e^{i\theta_{j+1}}$$

$$\Longrightarrow \quad e^{i\theta_j} = e^{i\theta_{j+1}} = e^{i\theta}$$

All exchanges are represented by the same phase, since they are indistinguishable.

Now, let us consider combination of anyons. Suppose we have an anyon with phase θ and we build a molecule from n of these anyons. What phase does one get by exchanging 2 such molecules?

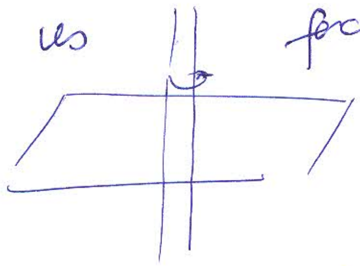
Each of the n charges in one molecule acquires a phase of $e^{i\theta/2}$ when transported half-way around each of the n fluxes in the other molecule. Hence the total phase generated is

$$e^{i 2n^2 \frac{\theta}{2}} = e^{i n^2 \theta}$$

This is also consistent with the spin-statistics theorem. Suppose we take a 2-anyon molecule, and rotate it thru' 2π (counter-clockwise.). Then besides the rotation of each of them, we also have one anyon revolving around the other (2 exchanges).

So the total phase $= e^{i\theta + i\theta + 2i\theta}$
$= e^{4i\theta} = e^{2^2 i\theta}$

Now, let us think of a more concrete model of an anyon. For this, we need to start with the Aharanov-Bohm effect. Let us consider a flux-tube $\perp^r$ to a plane and let us focus on the plane.



The flux-tube is just a small region here. If an electron is ~~taken~~ transported around the flux-tube, it acquires a phase $e^{iq\Phi}$ where Φ = flux and q = charge. The composite object formed by the flux and the charge (an electron stuck to the outside of the flux-tube) is the anyon. You can now think of rotating this counterclockwise object 'thru' 2π. The phase you would get

$$e^{\mp i 2\pi J} = e^{iq\Phi}$$

because the charge has gone around the flux.

$\theta = q\Phi = -2\pi J$ is the statistics parameter.